A tour in (formalised) type theory

Meven Lennon-Bertrand

Formalisation of mathematics with interactive theorem provers – 07/11/2024

✨ Formalised meta-theory of dependent types ✨

In particular:

- MetaCoq ≈ meta-theory of Coq, in Coq
- formalised logical relations

What do *you* know?

Let's start with a demo

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Summing up:

- coercions should be "transparent"!
- the current implementation does not deliver

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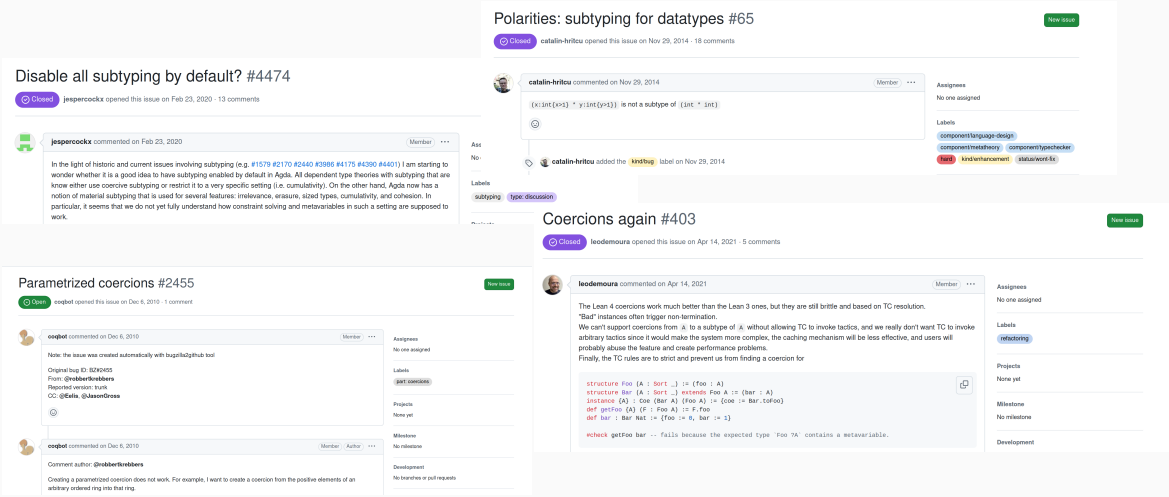

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Solutions?

- tactics, smart typeclasses...
- understand the type theory

Dependent type theory for the impatient

Martin-Löf's logical framework

Objects:

- types/propositions $A, B \ldots$
- terms/proofs $t, u \ldots$
- contexts/hypotheses $\Gamma, \ldots$ = lists $x_1 : A_1, \ldots, x_n : A_n$

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Judgements:

$$\Gamma \vdash A \qquad \Gamma \vdash A \cong B \qquad \Gamma \vdash t : A \qquad \Gamma \vdash t \cong u : A \qquad \vdash \Gamma \qquad \vdash \Gamma \cong \Delta$$

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Rules:

$$\frac{}{x_1 : A_1, \ldots x_n : A_n \vdash x_i : A_i} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \cong B}{\Gamma \vdash t : B}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t \cong t : A} \qquad \frac{\Gamma \vdash t \cong u : A \qquad \Gamma \vdash u \cong v : A}{\Gamma \vdash t \cong v : A} \qquad \ldots$$

Example: dependent functions

Lean: `(x : A) → B`, `fun x : A ⇒ t`, `t u`

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+ congruences

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Each time:

- types ($\Pi\, x{:}\, A.B$)
- introduction(s) ($\lambda\, x{:}\, A.B$)
- elimination(s) ($t\ u$)
- computation (β)
- uniqueness (η) (sometimes)

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Abstraction

Generalised algebraic theory:

$$\mathrm{Con} : \mathrm{Set} \qquad \mathrm{Ty} : \mathrm{Con} \to \mathrm{Set} \qquad \mathrm{Tm} : \Pi(\Gamma : \mathrm{Con}).\, \mathrm{Ty}(\Gamma) \to \mathrm{Set}$$

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The good properties

- Logical consistency: there are things we cannot prove
- Decidability: we can check that a proof is valid
- Model(s): we can interpret the language in some structure we like
- Expressivity: we can express our ideas “naturally”

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More technical:

- canonicity: every closed term is equal to a *value*

$$\cdot \vdash t : \mathrm{Bool} \quad \Rightarrow \quad \cdot \vdash t \cong \mathrm{true} : \mathrm{Bool} \quad \vee \quad \cdot \vdash t \cong \mathrm{false} : \mathrm{Bool}$$

- normalisation: every term is equal to a *normal form*

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- normalisation: every term is equal to a *normal form*
- subject reduction: program computation implies conversion

$$\Gamma \vdash t : T \quad \wedge \quad t \rightsquigarrow t' \quad \Rightarrow \quad \Gamma \vdash t \cong t' : T$$

Some meta-theory

How to prove consistency?

Build a model!

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Syntactically:

- Assume $\cdot \vdash t : \mathrm{False}$
- canonicity: there exists a value v such that $\cdot \vdash t \cong v : \mathrm{False}$
- but there are no values in False!

How to prove decidability?

$$\textsc{Conv}\ \frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \cong A'}{\Gamma \vdash t : A'}$$

Conv: what is the flow of information? → bidirectional typing

How to prove decidability?

$$\textsc{Conv}\ \frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \cong A'}{\Gamma \vdash t : A'} \qquad \textsc{Trans}\ \frac{\Gamma \vdash A \cong A' \qquad \Gamma \vdash A' \cong A''}{\Gamma \vdash A \cong A''}$$

Conv: what is the flow of information? → bidirectional typing

Trans: how do we guess A'?

Enter reduction

Naïve conversion checker:

$$\vdash A \cong B \quad \overset{?}{\Leftrightarrow} \quad A \rightsquigarrow \bar{A} \quad \wedge \quad B \rightsquigarrow \bar{B} \quad \wedge \quad \bar{A} \equiv_{\eta} \bar{B}$$

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- Does $\bar{A}$ always exist? (≈ normalisation)
  - progress: if $\Gamma \vdash A$, then either A is a normal form or $\exists A'.A \rightsquigarrow A'$
  - eventually, this terminates…

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Actually… 40+ years on doing this smartly!

How to prove subject reduction?

$$\Gamma \vdash (\lambda\, x{:}\, A.t)\, u : T \quad \Rightarrow \quad \begin{cases} \Gamma, x{:}\, A \vdash t : B \\ \Gamma \vdash u : A' \\ \Gamma \vdash \Pi\, x{:}\, A.B \cong \Pi\, x{:}\, A'.B' \\ \Gamma \vdash B'[x \leftarrow u] \cong T \end{cases}$$

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Injectivity of Π

How to prove progress?

$$\frac{\Gamma \vdash 0 : \mathrm{Nat} \qquad \Gamma \vdash u : A \qquad \Gamma \vdash \mathrm{Nat} \cong \Pi\, x{:}\, A.B}{\Gamma \vdash 0\; u : B[x \leftarrow u]}$$

should be impossible...

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No-confusion of type constructors

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Logical relations

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We need a predicate $\_ \Vdash \_ : \_$ that

- “satifies” the typing rules
- implies our desired property

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Core intuition for $\Gamma \Vdash t : A$

- t is good (here: normalises)
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- t can only be used in a good way

Roughly:

$$\Gamma \Vdash t : A \to B \coloneqq \begin{cases} t \text{ has a normal form } \bar{t} \\ \text{for any } u \text{ st. } \Gamma \Vdash u : A, \Gamma \Vdash \bar{t}\, u : B \end{cases}$$

But we are doing dependent types...

1. types are slippery (conversion)
2. universes: $\Gamma \Vdash t : \mathrm{Type} \coloneqq \Gamma \Vdash t ?$ $\lightning$
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1. $\Gamma \Vdash_{\mathcal{A}} t : A$ where $\mathcal{A} : \Gamma \Vdash A$
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And much more: Kripke quantification! Partial equivalence relations! Preasheaves! Gluing!

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- $\Gamma \vdash t : A \quad \Rightarrow \quad \Gamma \Vdash t : A \quad \Rightarrow \quad t$ normalises
- $\Gamma \vdash \mathrm{Nat} \cong \Pi\, x{:}\, A.B \quad \Rightarrow \quad \Gamma \Vdash \mathrm{Nat} \cong \Pi\, x{:}\, A.B \quad \Rightarrow \quad \lightning$
- $\Gamma \vdash \Pi\, x{:}\, A.B \cong \Pi\, x{:}\, A'.B' \quad \Rightarrow \quad \Gamma \Vdash \Pi\, x{:}\, A.B \cong \Pi\, x{:}\, A'.B'$
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  $\Rightarrow \quad \Gamma \Vdash A \cong A' \quad \Rightarrow \quad \Gamma \vdash A \cong A'$

Hurray!

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Inductive term : Type :=
  | tRel (n : nat)
  | tVar (id : ident) | tEvar (ev : nat) (args : list term)
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  | tSort (s : sort)
  | tProd (na : aname) (ty : term) (body : term)
  | tLambda (na : aname) (ty : term) (body : term)
  | tApp (u v : term)
  | tConst (c : kername) (u : Instance.t)
  | tInd (ind : inductive) (u : Instance.t)
  | tConstruct (ind : inductive) (idx : nat) (u : Instance.t)
  | tCase (ci : case_info) (type_info : predicate term)
      (discr : term) (branches : list (branch term))
  | tProj (proj : projection) (t : term)
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MetaCoq takes another way: confluence.

+ scales very well
+ injectivity/no-confusion without normalisation
- we don't know how to treat η...

Back to subtyping

What users™ want:

$$\textsc{Sub}\ \frac{\Gamma \vdash_{\text{sub}} t : A \qquad \Gamma \vdash_{\text{sub}} A \preccurlyeq A'}{\Gamma \vdash_{\text{sub}} t : A'}$$

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But it's a bit too wild...

Less frightening:

$$\textsc{Coe}\ \frac{\Gamma \vdash_{\mathrm{coe}} t : A \qquad \Gamma \vdash_{\mathrm{coe}} A \preccurlyeq A'}{\Gamma \vdash_{\mathrm{coe}} \mathrm{coe}_{A,A'}\, t : A'}$$

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- nicer semantics
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- more reasonable for implementations

This is work for a machine!

Let's elaborate, then:

$$\textsc{Sub}\ \frac{\Gamma \vdash_{\mathrm{sub}} t : A \qquad \Gamma \vdash_{\mathrm{sub}} A \preccurlyeq A'}{\Gamma \vdash_{\mathrm{sub}} t : A'} \quad \rightsquigarrow \quad \textsc{Coe}\ \frac{\tilde{\Gamma} \vdash_{\mathrm{coe}} \tilde{t} : \tilde{A} \qquad \tilde{\Gamma} \vdash_{\mathrm{coe}} \tilde{A} \preccurlyeq \tilde{A}'}{\tilde{\Gamma} \vdash_{\mathrm{coe}} \mathrm{coe}_{\tilde{A},\tilde{A}'}\, \tilde{t} : \tilde{A}'}$$

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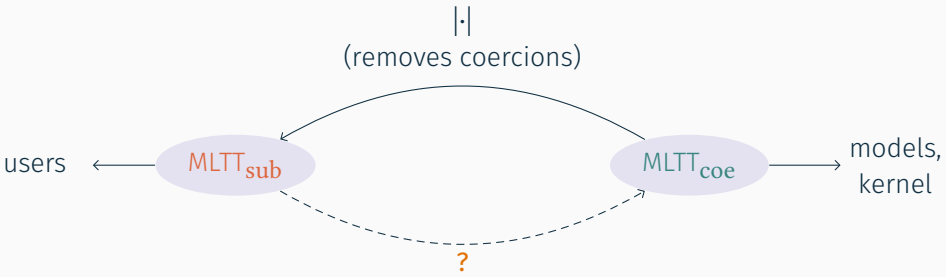
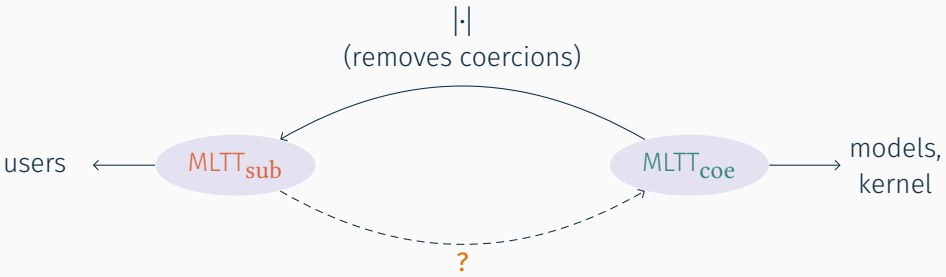
But subtyping is transitive:

$$\operatorname{coe}_{A',A''} \operatorname{coe}_{A,A'}\, t \overset{?}{\cong} \operatorname{coe}_{A,A''}\, t$$

Coherence



Coherence



? = the elaboration we want

Only well-defined if a lot of equations hold

In particular, $|t| = |u| \Rightarrow \Gamma \vdash_{\text{coe}} t \cong u : A$ (provided $\Gamma \vdash_{\text{coe}} t, u : A$)

A (vague) theorem

Laurent, L-B, Maillard (2024)

We can design a type theory $\mathrm{MLTT}_{\mathrm{coe}}$ that:

- keeps all the ✨good✨ properties
- is flexible enough to admit type-preserving elaboration from $\mathrm{MLTT}_{\mathrm{sub}}$

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The key: teach your kernel about definitional functoriality

$$\mathrm{map}\ \ g\ (\mathrm{map}\ \ f\ l) \cong \mathrm{map}\ (g \circ f)\ l$$

Thank you!